Set theory symbols are used to express relationships and operations between sets.

|  |  |  |
| --- | --- | --- |
| symbol |  | explanation |
|  | Subset | This symbol means “is a subset of”.  If , it means every element of set is also an element of set . It includes the possibility that could be equal to .  if and only if |
|  | Proper Subset | Similar to , but with the difference that  if , it means every element of set  is also an element of set ,  but is not equal to |
|  | Superset | This symbol means "is a superset of.  If , it means every element of set is also an element of set . It includes the possibility that could be equal to . |
|  | Proper Superset | Similar to , but with the difference that  if , it means every element of set  is also an element of set ,  but is not equal to |
|  | Not a Subset | This symbol indicates that a set is not a subset of another set.  For example, if , it means there exists at least one element in set that is not in set |
|  | Not a Superset | This symbol indicates that a set is not a superset of another set.  For example, if , it means there exists at least one element in set that is not in set . |
|  | Empty Set | This symbol represents the set with no elements, also known as the null set. |
|  | Union | The union of two sets and , denoted by , is the set containing all elements that are in or in , or in both. |
|  | Intersection | The union of two sets and , denoted by , is the set containing all elements that are in and |
|  | Set Difference | The set difference of two sets and , denoted by or , is the set containing all elements of that are not in . |

represents set membership

If is an element of , it doesn't necessarily mean that is a subset or superset of . It simply means that is one of the elements contained within the set .

Powersets

A power set is a mathematical concept that relates to sets and their subsets.

Given a set , the power set of , denoted by , is the set containing all possible subsets of , including the empty set and itself.

For example, let's say you have a set . The power set would contain all possible combinations of elements from , including the empty set and itself:

Where:

is the empty set

and are singleton sets containing only one element from .

is the set containing all elements of .

Collectively, these subsets make up the power set .

The cardinality (number of elements) of the power set of a set with elements is , including the empty set and the set itself.

Vector Spaces

**Vector Language: The axioms for a vector space**

A vector space is a set of vectors with addition and scalar multiplication that satisfy the axioms of addition and scalar multiplication.

In order for a set V to be a vector space the following 10 axioms must be true:

|  |  |  |
| --- | --- | --- |
| **A1** | **Closure under addition**  *For any vectors and in the set, is also in the set* |  |
| **A2** | **Existence of an additive identity**  *There exists a vector in the set such that for any vector in the set, .*  *Related:*  *- A3:Existence of additive inverses* |  |
| **A3** | **Existence of additive inverses**  *For every vector in the set, there exists a vector in the set such that* |  |
| **A4** | **Associativity of addition**  *For any vectors , , and in the set,*  *fundamental property of addition* |  |
| **A5** | **Commutativity of addition**  *For any vectors and* *in the set,*  *fundamental property of addition* |  |
| **M1** | **Closure under scalar multiplication**  *For any scalar and any vector in the set, is also in the set.*  *Implied by:*  *- A1:**Closure under addition* |  |
| **M2** | **Distributive property - vector addition**  *For any scalar and any vectors and in the set, +cv.*  *Implied by:*  *- A1:**Closure under addition*  *- A4:* *Associativity of addition*  *- A5:**Commutativity of addition* |  |
| **M3** | **Distributive Property - scalar addition**  *For any scalars and and any vector in the set, +cv.*  *Implied by:*  *- A1:**Closure under addition*  *- A4:* *Associativity of addition*  *- A5:**Commutativity of addition* |  |
| **M4** | **Associative Property**  (Compatibility of scalar multiplication with field multiplication)  *For any scalars and and any vector in the set, +cv.*  *Implied by:*  *- A1:**Closure under addition*  *- A4:* *Associativity of addition*  *- A5:**Commutativity of addition* |  |
| **M5** | **Multiplicative identity**  *For any vector in the set, , where 1 is the multiplicative identity of the underlying field*.  *Implied by:*  *- A1:**Closure under addition*  *- A4:* *Associativity of addition*  *- A5:**Commutativity of addition* |  |

**ASS2: [**Problem 5] (1)

*Determine whether each set equipped with the given operation is a vector space.*

*For those that are not vector space identify the vector space axioms that fail.*

*(1) The set with the standard operations on*

**Ref for proofs:** [**https://home.cc.umanitoba.ca/~borgerse/14R-T1/MATH2300/Worksheets/VectorSpaceExamples.pdf**](https://home.cc.umanitoba.ca/~borgerse/14R-T1/MATH2300/Worksheets/VectorSpaceExamples.pdf)

*(no Axioms* *failed for this question, this is just an example to show how to do this)*

|  |  |
| --- | --- |
| A1 | Let .  Then      Therefore A1 holds. |
| A2 | Let  the zero vector in be  Then      Therefore A2 holds. |
| A3 | Let  Then      Therefore A3 holds. |
| A4 | Addition in follows the same rules as addition in , so associativity holds.  OR  Let  Then |
| A5 | Addition in follows the same rules as addition in , so commutativity holds. |

All 5 fundamental axioms for vector spaces are satisfied,

equipped with the standard operations on  is indeed a vector space.

|  |  |
| --- | --- |
| **M1** | Let  Then      Therefore M1holds. |
| **M2** | Let  Then      Therefore M2holds. |
| **M3** | Let  Then            Therefore M3holds. |
| **M4** | Let  Then            Therefore M4holds. |
| **M5** | Let  Then        Therefore M5holds. |

**Subspaces**

Definition: A subset W of a vector space V over a field F is called a

subspace of V if W is a vector space over F with the operations of addition

and scalar multiplication defined on V.

A subset of is a subspace if it satisfies the following three properties:

**1. Contains the zero vector**: must contain the zero vector, denoted as **0**, which is the vector with all components equal to zero: (0,0,0,0).

2. **Closed under vector addition**: If **u** and **v** are vectors in , then **u**+**v** must also be in . In other words, the sum of any two vectors in must still be in .

**3. Closed under scalar multiplication**: If **u** is a vector in and *c* is a scalar, then *c***u** must also be in . In other words, multiplying any vector in by a scalar must still result in a vector in .

|  |  |  |
| --- | --- | --- |
| **A1** | **Closure under addition**  *For any vectors and in the set, is also in the set* |  |
| **A2** | **Existence of an additive identity**  *There exists a vector in the set such that for any vector in the set, .*  *Related:*  *- A3:Existence of additive inverses* |  |
| **M1** | **Closure under scalar multiplication**  *For any scalar and any vector in the set, is also in the set.*  *Implied by:*  *- A1:**Closure under addition* |  |

*These properties ensure that the subset behaves like a vector space in its own right, inheriting the vector addition and scalar multiplication properties of , and containing the zero vector, which is necessary for a vector space.*

*LINEAR COMBINATIONS AND SYSTEMS OF LINEAR EQUATIONS*

**Definition: linear combination**

*Let be a vector space over a field , and let , ​ be vectors in .*

*A linear combination of , ​ is any expression of the form:*

*Where are scalars from the field .*

*are the vectors being combined.*

*are the scalars that weight each vector.*

**Definition: Linear independence**

*Let be a vector space over a field , and let , ​ be vectors in .*

*Is said to be linearly independent if the only solution to the equation:*

*is the trivial solution,*

*where 0 is the zero vector in .*

Homogeneous Linear Systems

*A homogeneous linear system with coefficient matrix A has non-trivial solutions if and only if the columns of A are linearly dependent.*

Geometric interpretation of the set of solutions in

|  |  |  |  |
| --- | --- | --- | --- |
| ***geometric interpretation*** | *solution space* | *Nullity of A* | *Explanation* |
| **A line through the origin**: | one-dimensional. | 1 | looks like a straight line that passes through the origin (0, 0, 0) |
| **A plane through the origin**: | two-dimensional | 2 | looks like a flat plane that extends infinitely and passes through the origin |
| **The origin only**: | zero-dimensional | 0 | This means the solution space is zero-dimensional, consisting of only the origin itself.  The only solution to the system is the trivial solution . |

ASS4: Problem 13.

Find the coordinate vectors of v relative to the basis of set

, where

(a) ; ;

(b) ; ;

[1]

To find the coordinate vectors of relative to the basis set

, we can use the formula:

* Where the coordinate vector of relative to the basis set , is a column vector
* is the matrix formed by arranging ​ and as column vectors
* is the matrix formed by arranging the basis vectors as column vectors.

(a) Thus,

Therefore,

(b) Thus,

Therefore,

ASS4: Problem 14.

Let and be two subspaces of defined by

and

and

Find the bases of and

To find the bases of subspaces and , we'll follow these steps:

1. Express each subspace as the span of a set of vectors.
2. Apply the Gauss-Jordan elimination to find the basis vectors.

[1] For

We can rewrite the conditions as equations:

Now, let's express as the span of a set of vectors:

The vectors and are linearly independent and span , so they form a basis for .

[2] For

We can rewrite the conditions as equations:

Now, let's express as the span of a set of vectors:

The vectors and are linearly independent and span , so they form a basis for .

ASS4: Problem 15.

Determine whether the following form basis for :

(a) ; ;

(b) ; ;

To determine whether a set of polynomials forms a basis for , we need to check two conditions:

1. Linear Independence: The polynomials in the set must be linearly independent.
2. Spanning: The set must span , meaning that any polynomial in , can be expressed as a linear combination of the polynomials in the set.

(a) ; ;

[1] Express in terms of

Rewrite as system of equations:

Thus,

And,

And,

[2] Linearly independence

find constants , ​, and such that:

Rewrite as system of equations:

[1]

[2]

[3]

Thus,

Therefore, the polynomials are linearly independent.